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Multifractal analysis of the spectral measure of the Thue–Morse sequence: a periodic orbit approach

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Abstract

The Fourier spectral density of the Thue–Morse sequence is reinterpreted as the invariant measure of a stochastic dynamical system. Based on this fact, its generalized (Rényi) dimension and $f(\alpha)$ statistics are calculated with high precision by cycle expansions of spectral determinant and dynamical zeta function. α_q at integer values of q are also computed in an operator scheme and the asymptotic result in the large-q limit is derived.

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1. Introduction

The Thue–Morse sequence is one of the most famous automatic sequences [1, 2]. This infinite binary sequence is constructed by starting from the seed $K_0 = \{1\}$ and iteratively expanding the sequence according to the inflation rule, $1 \to 1, -1$ and $-1 \to -1, 1$, e.g. $K_1 = \{1, -1\}$, $K_2 = \{1 - 1, -1, 1\}$ and $K_3 = \{1 - 1, -1, 1, -1, 1, 1, -1\}$.

Being viewed as a prototype of a self-similar linear structure or time series which is neither regular nor random, the Thue–Morse sequence has been intensively studied in many fields of physics, such as solid-state physics, nonlinear dynamics and quantum chaos (see, for example, [3–9] and references therein). The marginal nature of the Thue–Morse sequence is best described by its Fourier spectral density, which is determined by $\rho(x) \equiv \lim_{n\to\infty} \rho^{(n)}(x)$, where $\rho^{(0)} = 1$ and

$$\rho^{(n+1)}(x) = (1 - \cos 2\pi x)\rho^{(n)}(2x). \tag{1.1}$$

Correspondingly, $\rho(x)$ is neither the sum of δ-functions (Bragg peaks) nor an ordinary continuous function. It produces a *singular continuous* measure instead [3].

Although the connection of peaks of $\rho(x)$ with periodic orbits of the Bernoulli map, $x \to 2x \pmod 1$, is rather evident, a quantitative study of this connection, especially from the viewpoint of nonlinear dynamics, is still lacking. In this paper we study the multifractal statistics of the Thue–Morse spectral density in the periodic orbit theory [10]. This paper is organized as follows. In section 2, we relate $\rho(x)$ to the equilibrium density of a stochastic dynamical system and its multifractal statistics to the leading eigenvalue of a Frobenius–Perron-like operator $\mathcal{L}_q$. In section 3 we calculate the eigenvalue by cycle expansions of the spectral determinant and dynamical zeta function. For comparison, we discuss briefly in section 4 the finite Markov chain approximation. Section 5 is devoted to the case when q is an integer, which reexplains and extends the results of [6]. We discuss the asymptotic behaviour when $q \to \infty$ in section 6, which is followed by a concise summary and discussion.

2. Stochastic model and multifractal analysis

We rewrite equation (1.1) as

$$\rho^{(n+1)}(x) = \int_0^1 \mathcal{L}(x, x')\rho^{(n)}(x')\,\mathrm{d}x' \equiv [\mathcal{L}\rho^{(n)}](x), \tag{2.1}$$

where

$$\begin{aligned}\mathcal{L}(x, x') &= (1 - \cos 2\pi x)[\delta(2x - x') + \delta(2x - x' - 1)] \\ &= \sin^2 \frac{\pi}{2} x'\delta\left(x - \frac{x'}{2}\right) + \cos^2 \frac{\pi}{2} x'\delta\left(x - \frac{x'+1}{2}\right).\end{aligned} \tag{2.2}$$

The form of $\mathcal{L}$ suggests that we can interpret it as the evolution operator of a stochastic dynamical system, in which x is mapped to $f_\sigma(x) = (x + \sigma)/2$ with probability $p_\sigma(x) = \frac{1}{2}[1 - (-1)^\sigma \cos \pi x]$ for $\sigma = 0, 1$. Equivalently, $\mathcal{L}$ describes a stochastic process of binary sequences,

$$\ldots s_2, s_1, s_0, s_{-1}, s_{-2}, \ldots$$

with conditional probability given by

$$P(s_{k+1}|s_k s_{k-1} s_{k-2} \ldots) = p_{s_{k+1}}(x_k), \tag{2.3}$$

where $s_j \in \{0, 1\}$ and $x_k = \sum_{j=0}^{\infty} s_{k-j} 2^{-j-1}$. According to this interpretation, $\rho(x)$ is the invariant density of the stochastic process and the probability of finding a segment of sequence is given by the integration of $\rho(x)$ over an interval, e.g.

$$P(0) = \int_0^{1/2} \rho(x)\,\mathrm{d}x = \frac{1}{2}, \qquad P(01) = \int_{1/4}^{1/2} \rho(x)\,\mathrm{d}x \approx 0.385\,169\,706\,934.$$

Obviously, $P(S)$ satisfies

$$P(S0) + P(S1) = P(0S) + P(1S) = P(S) \tag{2.4}$$

and

$$\sum_{|S|=l} P(S) = 1 \tag{2.5}$$

for an arbitrary binary sequence S and integer l, where $|S|$ denotes the length of S. The second part of equation (2.4) indicates that $\rho(x)$ is invariant under the Bernoulli map (shift).

$P(S)$ generally decreases exponentially with the increase of sequence length. Moreover, the exponents differ for different sequences. For example, $P((01)^l) \sim \left(\frac{3}{4}\right)^{2l}$, $P((0011)^l) \sim \left(\frac{\sqrt{5}}{4}\right)^{4l}$. One may ask what is the relative proportion of the sequences when they are categorized according to their decay exponents. This naturally leads to the multifractal analysis of the Thue–Morse spectral density, of which a basic assumption is that the number of sequences that decay as $P(S) \sim \left(\frac{1}{2}\right)^{\alpha|S|}$ is of the order of $2^{f(\alpha)|S|}$.

For a convenient calculation of $f(\alpha)$, one usually first considers the summation

$$\sum_{|S|=l} P^q(S) \sim \int \left(\frac{1}{2}\right)^{l(q\alpha - f(\alpha))} \mathrm{d}\alpha. \tag{2.6}$$

When $l \to \infty$, equation (2.6) implies

$$\sum_{|S|=l} P^q(S) \sim \left(\frac{1}{2}\right)^{l\tau_q}, \tag{2.7}$$

where

$$\tau_q = \min_{\alpha}\{q\alpha - f(\alpha)\} = q\alpha_q - f(\alpha_q) \tag{2.8a}$$

with α_q determined by $f'(\alpha_q) = q$. τ_q is related to the generalized (Rényi) dimension D_q by $D_q = \tau_q/(q-1)$. So $f(\alpha)$ can be obtained from τ_q by an inverse Legendre transformation, i.e.,

$$f(\alpha_q) = q\alpha_q - \tau_q, \tag{2.8b}$$

where $\alpha_q = \frac{\mathrm{d}\tau_q}{\mathrm{d}q}$.

In most cases, τ_q has to be acquired by a log–log fit according to its definition. However, in our case, it can be shown that

$$\tau_q = -\frac{\log \lambda_q}{\log 2}, \tag{2.9}$$

where λ_q is the leading eigenvalue of operator $\mathcal{L}_q$:

$$\mathcal{L}_q(x, x') = \mathrm{e}^{q \log \frac{1}{2}(1-\cos 2\pi x)} \sum_{\sigma} \delta(x - f_\sigma(x')) \equiv \mathrm{e}^{qA(x)} \sum_{\sigma} \delta(x - f_\sigma(x')) \tag{2.10}$$

(see appendix A). Moreover, according to the perturbation theory,

$$\frac{1}{\lambda_q}\frac{\mathrm{d}\lambda_q}{\mathrm{d}q} = \frac{1}{\lambda_q}\frac{\langle\phi_q|\frac{\partial}{\partial_q}\mathcal{L}_q|\rho_q\rangle}{\langle\phi_q|\rho_q\rangle} = \frac{1}{\lambda_q}\frac{\langle\phi_q|A\mathcal{L}_q|\rho_q\rangle}{\langle\phi_q|\rho_q\rangle} = \frac{\langle\phi_q|A|\rho_q\rangle}{\langle\phi_q|\rho_q\rangle} \tag{2.11}$$

where ρ_q (or ϕ_q) is the right (or left) eigenvector of $\mathcal{L}_q$ corresponding to eigenvalue λ_q and A is understood as a diagonal operator, i.e., $A(x, x') = A(x)\delta(x - x')$.

In this paper we shall calculate λ_q by cycle expansions and, for comparison, finite Markov chain approximation. Before doing this, let us consider two simple cases, which had been discussed in [3, 6]. One is $q = 0$. Since $\rho_0(x) = \phi_0(x) = 1$ and $\lambda_0 = 2$, we have $\tau_0 = -1$ and, according to equation (2.11),

$$\alpha_0 = -\frac{1}{\log 2}\int_0^1 A(x)\,\mathrm{d}x = 2. \tag{2.12}$$

Another is $q = 1$. Note that $\mathcal{L}_1 = \mathcal{L}$, $\lambda_1 = 1$, $\rho_1(x) = \rho(x)$ and $\phi_1(x) = 1$,

$$D_1 = \alpha_1 = -\frac{1}{\log 2}\int_0^1 A(x)\rho(x)\,\mathrm{d}x = 1 - \frac{1}{\log 2}\int_0^1 \log(1 - \cos 2\pi x)\rho(x)\,\mathrm{d}x. \tag{2.13}$$

Then D_1 can be calculated to very high precision by making use of a technique of convergence acceleration. In this paper, we also generalize this technique to the case when q is an arbitrary positive integer.

3. Cycle expansion

An idea of great importance in chaotic dynamics is that *the spectrum of eigenvalues is dual to the spectrum of periodic orbits* [11]. Although the reason why the eigenvalues of an operator supported by chaotic dynamics can be efficiently extracted from the information of periodic orbits is physically profound and mathematically hard, the calculation is, at least in our case, rather standard. In the following we give a self-contained description of the computation scheme. For a more detailed discussion of the periodic orbit theory, we refer readers to the web book [11].

We first reduce the system according to the reflection symmetry, $x \leftrightarrow 1-x$. By identifying x with $1-x$, we restrict $x \in \left[0, \frac{1}{2}\right]$ and rewrite $\mathcal{L}_q(x, x')$ as

$$\mathcal{L}_q(x, x') = \mathrm{e}^{qA(x)}[\delta(x - f_+(x')) + \delta(x - f_-(x'))], \tag{3.1}$$

where

$$f_+(x) = \frac{x}{2} \qquad \text{and} \qquad f_-(x) = \frac{1-x}{2} \tag{3.2}$$

are the two branches of the inverse of the Baker's map.

Then consider the trace

$$\mathrm{tr}\big(\mathcal{L}_q^t\big) = \int_0^{\frac{1}{2}} \mathcal{L}_q^t(x, x)\,\mathrm{d}x, \tag{3.3}$$

$t = 1, 2, \ldots$. This leads to the trace formula

$$\mathrm{tr}\left(\frac{z\mathcal{L}_q}{1 - z\mathcal{L}_q}\right) = \sum_{t=1}^{\infty} z^t\,\mathrm{tr}\big(\mathcal{L}_q^t\big) = \sum_p \sum_{r=1}^{\infty} \frac{n_p\,\mathrm{e}^{rqA_p} z^{rn_p}}{1 - d_p^r}, \tag{3.4}$$

where p denotes a prime periodic orbit of the Baker's map. n_p, A_p and d_p are explained as follows. It is well known that the periodic orbits of the Baker's map are coded by binary sequences. For $\Sigma = \sigma_k \cdots \sigma_2\sigma_1$, $\sigma_i \in \{+, -\}$, define

$$f_\Sigma(x) \equiv f_{\sigma_k} \circ \cdots f_{\sigma_2} \circ f_{\sigma_1}(x). \tag{3.5}$$

Equation (3.3) is evaluated at each fixed point of f_Σ, $|\Sigma| = t$, which can be arranged into the sum over prime periodic orbits whose periods are divisors of t. For a prime periodic orbit coded by Σ, $n_p = |\Sigma|$, $d_p = f'_\Sigma = \left(\frac{1}{2}\right)^{n_p}$ (or $-(\frac{1}{2})^{n_p}$) if the number of '$-$' contained in Σ is even (or odd) and A_p is the sum of $A(x)$ over the periodic orbit. For example,

$$\begin{aligned}
&\mathrm{e}^{A_+} = \mathrm{e}^{A(0)} = 0,\\
&\mathrm{e}^{A_-} = \mathrm{e}^{A(1/3)} = 3/4,\\
&\mathrm{e}^{A_{+-}} = \mathrm{e}^{A(1/5)+A(2/5)} = 5/16,\\
&\mathrm{e}^{A_{++-}} = \mathrm{e}^{A(1/9)+A(2/9)+A(4/9)} = 3/64,\\
&\mathrm{e}^{A_{+--}} = \mathrm{e}^{A(1/7)+A(2/7)+A(3/7)} = 7/64.
\end{aligned}$$

The trace formula diverges at the reciprocal of each eigenvalue of $\mathcal{L}_q$. This fact enables us to determine λ_q from its convergent radius. A more powerful method is to find the smallest zero of the spectral determinant $\det(1 - z\mathcal{L}_q)$, which is formally related to the trace formula by

$$-z\frac{\partial}{\partial z}\log\det(1 - z\mathcal{L}_q) = \mathrm{tr}\left(\frac{z\mathcal{L}_q}{1 - z\mathcal{L}_q}\right), \tag{3.6}$$

or the dynamical zeta function $1/\zeta(z)$,

Table 1. τ_q and α_q calculated by cycle expansions and finite Markov chain approximation (FMCA). k in the first column denotes the maximal cycle length or the order of Markov chain. The 'exact' values of τ_q and α_q are calculated by the cycle expansion of the spectral determinant at $k = 10$.

$q = 0.5$	τ_q			α_q		
k	$\det(1 - z\mathcal{L}_q)$	$1/\zeta(z)$	FMCA	$\det(1 - z\mathcal{L}_q)$	$1/\zeta(z)$	FMCA
1	–	–	−0.386	0.4	0.4	1
2	−0.28	−0.38	−0.397	0.6	0.6	1.0
3	−0.395	−0.405	−0.402	0.84	0.90	0.94
4	−0.403 1	−0.403 3	−0.403 1	0.904	0.910	0.915
5	−0.403 423	−0.403 43	−0.403 3	0.908 28	0.908 0	0.910
6	−0.403 426 10	−0.403 425 8	−0.403 40	0.908 374 4	0.908 4	0.909
7	−0.403 426 114 58	−0.403 426 116	−0.403 421	0.908 374 996	0.908 370	0.9085
Exact	−0.403 426 114 600 98 . . .			0.908 374 997 952 66 . . .		

$$1/\zeta(z) = \Pi_p(1 - z^{n_p}\, e^{qA_p}). \tag{3.7}$$

The dynamical zeta function is an approximation of the spectral determinant if d_p in the trace formula is omitted.

Based on the data of all periodic orbits whose lengths do not exceed an integer k, the trace formula is calculated as a polynomial of order k, from which the spectral determinant is obtained according to equation (3.6). A crucial step is that the spectral determinant is also expanded as a k-order polynomial of z. This truncation is reasonable since the higher order terms will be effectively cancelled by the contributions of the longer periodic orbits which have not yet been included. Similarly, the dynamical zeta function is constructed as a k-order polynomial from the same set of periodic orbits. For example, when $k = 3$,

$$\mathrm{tr}\left(\frac{z\mathcal{L}_q}{1 - z\mathcal{L}_q}\right) \approx \frac{1}{2}\left(\frac{3}{4}\right)^{q-1} z + \frac{1}{4}\left[3\left(\frac{9}{16}\right)^{q-1} + 2\left(\frac{5}{16}\right)^{q-1}\right] z^2 + \frac{1}{8}\left[3\left(\frac{27}{64}\right)^{q-1} + 3\left(\frac{7}{64}\right)^{q-1} + \left(\frac{3}{64}\right)^{q-1}\right] z^3,$$

$$\det(1 - z\mathcal{L}_q) \approx 1 - \frac{1}{2}\left(\frac{3}{4}\right)^{q-1} z - \frac{1}{4}\left[\left(\frac{9}{16}\right)^{q-1} + \left(\frac{5}{16}\right)^{q-1}\right] z^2 - \frac{1}{8}\left[\left(\frac{7}{64}\right)^{q-1} + \frac{1}{3}\left(\frac{3}{64}\right)^{q-1} - \left(\frac{15}{64}\right)^{q-1} - \frac{1}{3}\left(\frac{27}{64}\right)^{q-1}\right] z^3$$

and

$$1/\zeta(z) \approx 1 - \left(\frac{3}{4}\right)^{q} z - \left(\frac{5}{16}\right)^{q} z^2 - \left[\left(\frac{7}{64}\right)^{q} + \left(\frac{3}{64}\right)^{q} - \left(\frac{15}{64}\right)^{q}\right] z^3.$$

Finally, $z_q = 1/\lambda_q$ is obtained by numerically solving the smallest zero of a polynomial $H(z)$ and its derivative is calculated according to

$$\frac{1}{\lambda_q}\frac{\partial \lambda_q}{\partial q} = -\frac{1}{z_q}\frac{\partial z_q}{\partial q} = \frac{1}{z_q}\frac{\partial H}{\partial q}\bigg/\frac{\partial H}{\partial z}\bigg|_{z=z_q}. \tag{3.8}$$

The convergence of τ_q and α_q is typically very fast. For the dynamical zeta function, it is exponential while for spectral determinant, it is even faster (see table 1). This is in accordance

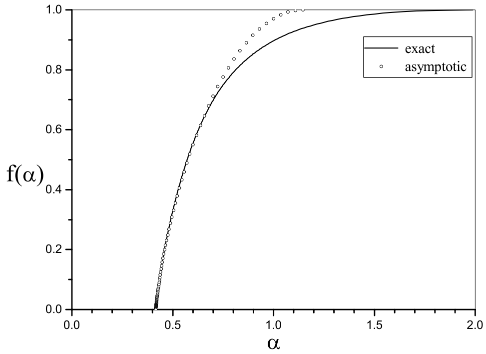

Figure 1. $f(\alpha)$ curve for the Thue–Morse sequence. Note that the right part of this curve is absent due to the singularity of $A(x)$ at $x = 0$ [3]. The asymptotic curve is calculated according to $\tau_q \approx (\frac{3}{4})^q(1+\gamma_+^q)$ (see equation (6.4)).

with results in a general chaotic system with complete symbolical dynamics [11]. The $f(\alpha)$ curve is shown in figure 1. Here we point out two notable phenomena.

One is that the convergence of spectral determinant becomes poor if q is too close to 0. This is related to the singularity of $A(x)$ at $x = 0$. In fact, if $A(x)$ is bounded, then

$$\operatorname{tr}\left(\frac{z\mathcal{L}_0}{1-z\mathcal{L}_0}\right) = \sum_{k=1}^{\infty} 2^{k-1} z^k \left(\frac{1}{1-2^{-k}} + \frac{1}{1+2^{-k}}\right) = \sum_{k=1}^{\infty} \frac{(2z)^k}{1-2^{-2k}}, \tag{3.9}$$

which implies

$$\det(1 - z\mathcal{L}_0) = \Pi_{j=0}^{\infty}(1 - 2^{1-2j} z) \tag{3.10}$$

is an entire function of z and its coefficient c_k of series expansion decays faster than any power of k. However, $A(0) = -\infty$ in our case, and hence

$$\operatorname{tr}\left(\frac{z\mathcal{L}_0}{1-z\mathcal{L}_0}\right) = \sum_{k=1}^{\infty} z^k \left(\frac{2^{k-1}-1}{1-2^{-k}} + \frac{2^{k-1}}{1+2^{-k}}\right) = \sum_{k=1}^{\infty} \frac{(2z)^k}{1-2^{-2k}} - \frac{z^k}{1-2^{-k}}, \tag{3.11}$$

which implies that

$$\det(1 - z\mathcal{L}_0) = \frac{\Pi_{j=0}^{\infty}(1 - 2^{1-2j} z)}{\Pi_{j=0}^{\infty}(1 - 2^{-j} z)} = \frac{(1-2z)}{\Pi_{j=0}^{\infty}(1 - 4^{-j} z)} \tag{3.12}$$

is no longer an entire function. Instead, it has poles at $z = 4^j$, $j = 0, 1, \ldots,$ and, consequently, c_k tends to a finite number when $k \to \infty$ and the convergence of z_0 is only as 2^{-k}. From the viewpoint of continuity, this singularity will affect the rate of convergence at small q.

Another interesting fact is that, if q is an integer, the spectral determinant produces the exact λ_q (or τ_q) when $k \geqslant q + 1$ (for example, see table 2). This fact implies that $\det(1 - z\mathcal{L}_q)$ is a polynomial of order $q + 1$. We shall discuss this fact in more detail in section 5.

4. Finite Markov chain approximation

Owing to its conceptual simplicity and numerical facility, approximating a stochastic process by a Markov chain of finite order is always a physically appealing approach. Once the approximated transfer matrix is obtained (see appendix A), we calculate its leading eigenvalue

Table 2. Same as table 1. The exact value of τ_q is $3 - \log(1+\sqrt{17})/\log 2$ [5].

$q = 2$	τ_q			α_q		
k	$\det(1 - z\mathcal{L}_q)$	$1/\zeta(z)$	FMCA	$\det(1 - z\mathcal{L}_q)$	$1/\zeta(z)$	FMCA
1	1.4	0.8	0.4	0.4	0.4	1
2	0.53	0.51	0.69	0.51	0.55	0.7
3	0.642 981 363 139 42	0.71	0.61	0.56	0.58	0.6
4	0.642 981 363 139 42	0.62	0.65	0.574 58	0.566	0.59
5	0.642 981 363 139 42	0.651	0.638	0.574 665 7	0.581	0.5739
6	0.642 981 363 139 42	0.640	0.644	0.574 666 108	0.572	0.5766
7	0.642 981 363 139 42	0.644	0.643	0.574 666 109 1368	0.576	0.5741
Exact	0.642 981 363 139 42 ...			0.574 666 109 137 05 ...		

by using the well-known power method. As a byproduct, we obtain also its right and left eigenvectors, which can be used to estimate α_q according to equation (2.11).

The convergence of the finite Markov chain approximation (FMCA) is a little slower than that of the dynamical zeta function if we identify the order of Markov chain with the maximal cycle length. (Of course, the numerical costs of the two methods at the same level of approximation are not identical.) Compared with the cycle expansion of the spectral determinant, the rate of convergence is, especially when q is large, rather poor. However, FMCA is more informative since it provides also the eigenfunctions. It should be pointed out that since $\rho_q(x)$ is a generalized function, we cannot expect, for almost every point $x \in [0, 1]$, a well-defined limit of $\rho_q(x)$ when the order of Markov chain goes to infinity. Instead, we can only obtain vectors with violent fluctuations on an increasingly small scale. Moreover, in the coarse-grained picture we generally have $\rho_q(x) \neq c[\rho(x)]^q$. Several $\rho_q(x)$ and $\phi_q(x)$ are shown in figure 2, from which we can see that, with the increase of q, $\rho_q(x)$ becomes more and more localized at the periodic orbit $\left\{\frac{1}{3}, \frac{2}{3}\right\}$ and its stable manifold while the smoothness of $\phi_q(x)$ keeps unchanged.

5. Case of integer *q*

It was found that τ_q for integer values of q can be related to the leading eigenvalue of a finite dimensional, $q \times q$ in fact, matrix [6]. This matrix, as we shall show, naturally arises when we restrict $\mathcal{L}_q^\dagger$ to a finite invariant subspace. Note that

$$\left[\mathcal{L}_q^\dagger f\right](x) = \sin^{2q}\left(\frac{\pi}{2}x\right) f\left(\frac{x}{2}\right) + \cos^{2q}\left(\frac{\pi}{2}x\right) f\left(\frac{x+1}{2}\right), \tag{5.1}$$

in the symmetric space with basis $\{e_n = \cos(2n\pi x)\}_{n\geqslant 0}$, the action of $\mathcal{L}_q^\dagger$ is represented by

$$\mathcal{L}_q^\dagger e_{2n} = \frac{1}{2^{2q-1}} \sum_{k=-[\frac{q}{2}]}^{[\frac{q}{2}]} \frac{(2q)!}{(q-2k)!(q+2k)!} e_{|n+k|} \tag{5.2}$$

and

$$\mathcal{L}_q^\dagger e_{2n+1} = -\frac{1}{2^{2q-1}} \sum_{k=-[\frac{q-1}{2}]}^{[\frac{q+1}{2}]} \frac{(2q)!}{(q-2k+1)!(q+2k-1)!} e_{|n+k|}. \tag{5.3}$$

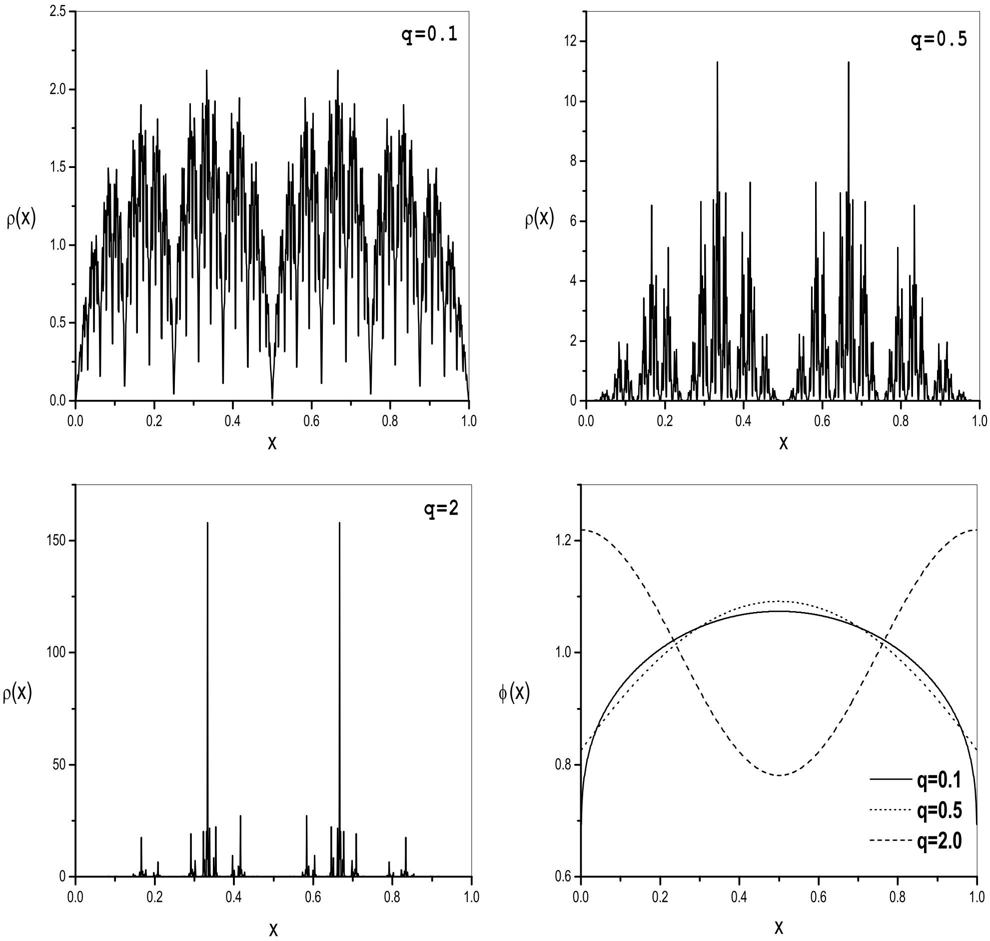

Figure 2. Some ρ_q and ϕ_q calculated by Markov chain of order 10. The eigenfunctions plotted are scaled to $\int_0^1 \rho_q(x) = \int_0^1 \phi_q(x) = 1$.

It can be easily verified that $\{e_0, e_1, \ldots, e_q\}$ span an invariant subspace of $\mathcal{L}_q^\dagger$, which we denote by $\mathcal{S}_q$. Moreover, $\mathcal{S}_q$ is a *sink* in the sense that for any $e_n \notin \mathcal{S}_q$ we have $\mathcal{L}_q^{\dagger k} e_n \in \mathcal{S}_q$ if k is sufficiently large. One may infer from this fact that the non-zero component of the spectrum of $\mathcal{L}_q^\dagger$ is given by its restriction within $\mathcal{S}_q$, which we denote by $\mathcal{M}_q$, e.g.

$$\mathcal{M}_1 = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}, \quad \mathcal{M}_2 = \frac{1}{8}\begin{bmatrix} 6 & -4 & 1 \\ 2 & -4 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{M}_3 = \frac{1}{32}\begin{bmatrix} 20 & -15 & 6 & -1 \\ 12 & -16 & 20 & -15 \\ 0 & -1 & 6 & -15 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Obviously, the leading eigenvalue of $\mathcal{M}_q$ is determined by its first $q \times q$ block, which is, up to a constant, identical with the matrix constructed in [6]. Furthermore, assume that $\mathcal{L}_q$ and $\mathcal{L}_q^\dagger$ have the same spectrum, then we have

$$\det(1 - z\mathcal{L}_q) = \det(1 - z\mathcal{M}_q) \tag{5.4}$$

is a $(q + 1)$-order polynomial of z. We give a rigorous proof of equation (5.4) in appendix B, where a counter example is also presented to show the insufficiency of the above reasoning. We point out that a similar result, i.e., the spectral determinant is given by a polynomial, has also been established in some systems by quite different methods [18].

Then we consider the eigenfunctions. Denote the right and left eigenvectors of $\mathcal{M}_q$ corresponding to the leading eigenvalue by $(u_0, u_1, \ldots, u_q)^T$ and $(v_0, v_1, \ldots, v_q)$ respectively, then

$$\phi_q = \sum_{j=0}^{q} u_j e_j \qquad \text{and} \qquad \rho_q = \sum_{j=0}^{q} v_j e_j^* + \sum_{j=q+1}^{\infty} c_j e_j^* \equiv \rho_q^{(0)} + \rho_q', \tag{5.5}$$

where e_j^* is the dual state of e_j, i.e., $e_0^* = e_0$ and $e_j^* = 2e_j$ for $j > 0$. ϕ_q and ρ_q are scaled to $\langle\phi_q|\rho_q\rangle = \sum_j u_j v_j = 1$ and c_j's can be determined by the relation

$$c_j = \langle e_j|\rho_q\rangle = \frac{1}{\lambda_q^k}\langle e_j|\mathcal{L}_q^k|\rho_q\rangle = \frac{1}{\lambda_q^k}\big\langle \mathcal{L}_q^{\dagger k} e_j\big|\rho_q^{(0)}\big\rangle, \tag{5.6}$$

where k is an integer that ensures $\mathcal{L}_q^{\dagger k} e_j \in \mathcal{S}_q$.

Equation (5.6) can be refined to an efficient algorithm to calculate the average of $f(x)$ with respect to the density given by $\rho_q(x)$. By iterating the procedure

$$\begin{aligned}\langle f|\rho_q\rangle &= \langle \mathcal{P}_q f|\rho_q\rangle + \langle (1-\mathcal{P}_q) f|\rho_q\rangle \\ &= \big\langle f\big|\rho_q^{(0)}\big\rangle + \frac{1}{\lambda_q}\big\langle \mathcal{L}_q^{\dagger}(1-\mathcal{P}_q) f\big|\rho_q\big\rangle,\end{aligned} \tag{5.7}$$

where $\mathcal{P}_q = \sum_{j=0}^{q} |e_j\rangle\langle e_j^*|$ is the projection operator of $\mathcal{S}_q$, we have

$$\langle f|\rho_q\rangle = \sum_{k=0}^{\infty}\big\langle f^{(k)}\big|\rho_q^{(0)}\big\rangle \tag{5.8a}$$

where

$$f^{(k)} = \left[\frac{1}{\lambda_q}\mathcal{L}_q^{\dagger}(1-\mathcal{P}_q)\right]^k f. \tag{5.8b}$$

Equation (5.8) converges very fast if $f(x)$ is a smooth function in $(0, 1)$. To get this excellent convergence, let us calculate $\alpha_q = \langle f|\rho_q\rangle$, where

$$f(x) = -\frac{1}{\log 2}A(x)\phi_q(x) = 2\phi_q(x) + \frac{1}{\log 2}\sum_{j=1}^{\infty}\sum_{l=0}^{q}\frac{u_l}{j}[e_{j+l}(x) + e_{|j-l|}(x)].$$

The results for several q's are listed in table 3, from which we can see that, especially for large q, the convergence is even faster than the cycle expansion of the spectral determinant.

6. Large-q limit

As suggested by the behaviour of ρ_q, with the increase of q, the behaviour of $\mathcal{L}_q$ is more and more dominated by the periodic orbit $\left\{\frac{1}{3}, \frac{2}{3}\right\}$ with code '$-$'. In fact, it was found that $\lambda_q \to \mathrm{e}^{qA_-} = \left(\frac{3}{4}\right)^q$ when $q \to \infty$ [6]. In this section we derive a higher order asymptotic correction. For convenience, we scale $\mathcal{L}_q$ to $\tilde{\mathcal{L}}_q = \left(\frac{4}{3}\right)^q \mathcal{L}_q$, or, equivalently, $A(x)$ to $\tilde{A}(x) = A(x) - A\left(\frac{1}{3}\right)$.

The fast convergence of the cycle expansion of the dynamical zeta function is guaranteed by the fact that the contribution of a periodic orbit is, on the whole, determined by its length.

Table 3. τ_q calculated according to equation (5.8). Comparing it with the cycle expansion of the spectral determinant, we can see that the two methods yield the identical 14 digits.

	$\langle f^{(k)}\vert\rho_q^{(0)}\rangle$			
k	$q=2$	$q=3$	$q=4$	$q=5$
0	0.057 802 330 305 33	1.004 936 532 649 72	0.456 674 436 625 05	0.021 728 026 996 66
1	0.515 190 258 970 08	−0.517 318 288 473 71	−0.013 486 500 813 89	0.403 444 068 082 03
2	0.001 651 036 299 67	0.000 072 274 766 79	−0.000 001 565 961 87	−0.000 000 124 377 09
3	0.000 022 325 014 79	−0.000 000 211 560 57	−0.000 000 001 787 65	0.000 000 000 026 69
4	0.000 000 158 214 87	0.000 000 000 348 26	−0.000 000 000 000 90	–
5	0.000 000 000 332 15	−0.000 000 000 000 12	–	–
6	0.000 000 000 000 15	–	–	–
α_q	0.574 666 109 137 05	0.487 690 307 730 38	0.443 186 368 060 74	0.425 171 970 728 28

However, for an intermittent system where the dynamics is strongly influenced by a marginal stable periodic orbit, the cycle expansion according to the cycle length generally converges very slow. To deal with this difficulty, one must construct a resummation scheme for the cycle expansion based on the cycle contributions (see, for example, [11–16]). This is what occurred in our case. Since the contribution of a periodic orbit is largely controlled by the number of '+' contained in its code sequence, we rewrite the symbolic code as $(j) \equiv +-^j$, $(jk) \equiv +-^j+-^k$ and so on. Then the dynamical zeta function is expanded according to the length of the new symbolic sequence, i.e.,

$$\begin{aligned}1/\zeta(z) = \Pi_p(1 - z^{n_p}\,\mathrm{e}^{q\tilde{A}_p}) = (1-z)\Bigg\{1 - \sum_j [j]z^{j+1} - \sum_{j<k}([jk]-[j][k])z^{j+k+2} \\ - \sum_{j<k<l}([jkl]+[jlk]-[j][kl]-[k][jl]-[l][jk]+[j][k][l])z^{j+k+l+3} \\ - \sum_{j\neq k}([jkk]-[k][jk])z^{j+2k+3} - \cdots\Bigg\},\end{aligned} \tag{6.1}$$

where $[j] \equiv \mathrm{e}^{q\tilde{A}_{(j)}}$, $[jk] \equiv \mathrm{e}^{q\tilde{A}_{(jk)}}$, $[jkl] \equiv \mathrm{e}^{q\tilde{A}_{(jkl)}}$, etc. Note that $[k_1k_2] \sim [k_1][k_2]$, $[k_1k_2k_3] \sim [k_1][k_2][k_3], \ldots,$ for sufficiently large k_i's, good cancellation can be expected in expansion (6.1). Therefore, an appropriate truncation of this expansion may yield a reasonable asymptotic expression of the zero of $1/\zeta(z)$. For example, when the first two items of equation (6.1) are retained, we have

$$1/\zeta(z) \approx (1-z)\left(1 - \sum_{j=0}^{\infty}[j]z^{j+1}\right) = (1-z)\left[1 - \sum_{j=0}^{\infty}\left([j]-\gamma_+^q\right)z^{j+1}\right] - \gamma_+^q z = 0, \tag{6.2}$$

where

$$\gamma_+ \equiv \lim_{j\to\infty} \mathrm{e}^{\tilde{A}_{(j)}} = 0.363\,247\,682\,720\,01\ldots, \tag{6.3}$$

which implies $z_q \sim 1 - \gamma_+^q$, or

$$\lambda_q \sim \left(\tfrac{3}{4}\right)^q\left(1+\gamma_+^q\right). \tag{6.4}$$

Similarly, if one more term in equation (6.1) is included, we have

$$\lambda_q \sim \left(\tfrac{3}{4}\right)^q\left(1+\gamma_+^q+\gamma_{+-+}^q\right), \tag{6.5}$$

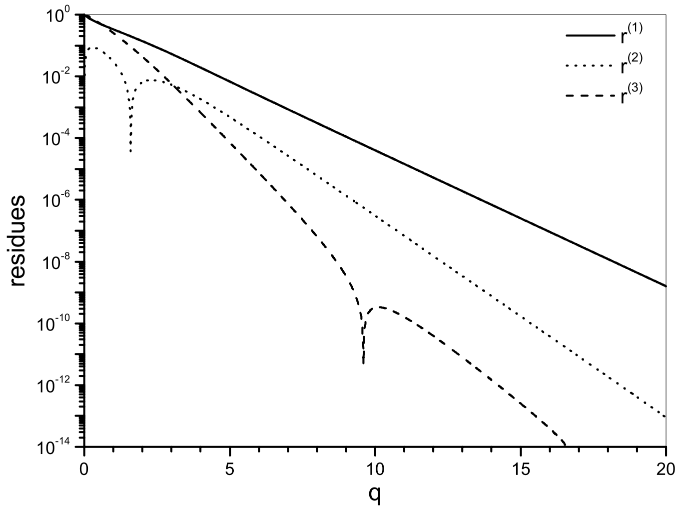

Figure 3. Asymptotic behaviour of λ_q, $r^{(1)} \equiv |(\frac{4}{3})^q\lambda_q - 1|$, $r^{(2)} \equiv |(\frac{4}{3})^q\lambda_q - 1 - \gamma_+^q|$, $r^{(3)} \equiv |(\frac{4}{3})^q\lambda_q - 1 - \gamma_+^q - \gamma_{+-+}^q|$ (see equation (6.5)).

where

$$\gamma_{+-+} \equiv \lim_{j\to\infty} \mathrm{e}^{\tilde{A}(1j)} = 0.223\,172\,892\,216\,68\ldots. \tag{6.6}$$

(For numerical verification of the asymptotic result, see figure 3.)

7. Summary and discussion

In this paper we first interpret the Fourier spectral density of the Thue–Morse sequence as the equilibrium state of a stochastic dynamical system. Then its multifractal statistics, τ_q, is determined by the leading eigenvalue of a Frobenius–Perron-like operator $\mathcal{L}_q$, which has been calculated by cycle expansions and, for comparison, finite-order Markov chain approximation. The cycle expansion of the spectral determinant converges very fast unless q is too close to 0, where the convergence is affected by poles. In the case of q being an integer, it is found that $\det(1 - z\mathcal{L}_q)$ is a polynomial of order $q + 1$ and the nonzero part of the spectrum of $\mathcal{L}_q$ is given by a $(q + 1) \times (q + 1)$ matrix, which is obtained by restricting $\mathcal{L}_q^\dagger$ within a $(q + 1)$-dimensional invariant subspace. Moreover, a convergence acceleration algorithm to calculate α_q is generalized to this case. Finally, by expanding the dynamical zeta function in a new symbolic scheme based on cycle contributions, we derive an asymptotic expression of τ_q in the large q limit.

We have studied the statistical property of the Thue–Morse sequence by cycle expansions in the periodic orbit theory. The method is powerful and, of course, not restricted to this problem [11]. However, there are some open questions. The most interesting one is what happens when q is close to an integer. For example, does $\mathcal{L}_q$ have finite number of nonzero eigenvalues? If it does, how does it change when q runs from k to $k + 1$? Or, if not, how do the many zeros of the spectral determinant collapse when $q \to k$? We note that the left eigenfunction of $\mathcal{L}_q$ can be extended to a smooth function on S^1, i.e., when 1 is identified with 0, only if q is an integer. This fact may be important to the answer. Another question concerns the asymptotic expansion of λ_q. It is interesting to note that, although we start from periodic orbits, the final result is in fact expressed in terms of homoclinic orbits. It is not clear whether similar result can be established in general chaotic systems with intermittent dynamics.

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Appendix A. Multifractal analysis based on Markov model

In this appendix we give a proof of equation (2.9). Since the dependence of s_{k+1} upon s_{k-j} decays as 2^{-j}, we can approximate the stochastic process by a Markov chain of finite order, i.e.,

$$P(s_{k+1}|s_k s_{k-1} s_{k-3} \ldots) \approx P^{(k)}(s_{k+1}|s_k s_{k-1} s_{k-3} \ldots s_1), \tag{A.1}$$

for sufficiently large k. The transfer probability $P^{(k)}(\sigma|S)$ can be evaluated from $p_\sigma(x)$, for example, at the middle point of the corresponding interval. For $S = s_k s_{k-1} \cdots s_1$, denote $\sigma s_{k-1} s_{k-2} \cdots s_2$ by $\mathcal{F}_\sigma S, \sigma = 0, 1$. Then the finite transfer matrix is given by

$$T(S', S) = \begin{cases} P^{(k)}(\sigma|S) & \text{if } S' = \mathcal{F}_\sigma S \\ 0 & \text{otherwise.} \end{cases} \tag{A.2}$$

Accordingly, the probability to find a very long sequence $\sigma_j \sigma_{j-1} \cdots \sigma_1 S_0$ is given by

$$P(\sigma_j \sigma_{j-1} \cdots \sigma_1|S_0) P(S_0) = T(S_j, S_{j-1}) T(S_{j-1}, S_{j-2}) \cdots T(S_1, S_0) P(S_0), \tag{A.3}$$

where $S_1 = \mathcal{F}_{\sigma_1} S_0$, $S_2 = \mathcal{F}_{\sigma_2} S_1, \ldots S_j = \mathcal{F}_{\sigma_j} S_{j-1}$. Consider

$$\sum_{|S'|=j+k} P^q(S') = \sum_{\sigma_j, \sigma_{j-1} \cdots \sigma_1} \sum_{S_0} P^q(\sigma_j \sigma_{j-1} \cdots \sigma_1 | S_0) P^q(S_0). \tag{A.4}$$

The summation over σ_i's can be replaced by that over S_i's, i.e.,

$$\begin{aligned} \sum_{|S'|=j+k} P^q(S') &= \sum_{S_j, S_{j-1} \ldots S_0} [T(S_j, S_{j-1}) T(S_{j-1}, S_{j-2}) \cdots T(S_1, S_0) P(S_0)]^q \\ &= \sum_{S_j, S_0} \langle S_j | T_q^j | S_0 \rangle P^q(S_0) \sim \lambda_q^j, \qquad (j \to \infty) \end{aligned} \tag{A.5}$$

where the matrix T_q is defined according to

$$T_q(S', S) = \begin{cases} [T(S', S)]^q & \text{if} \quad T(S', S) > 0 \\ 0 & \text{if} \quad T(S', S) = 0 \end{cases} \tag{A.6}$$

and λ_q is its leading eigenvalue. Note that T_q is the discrete version of the operator $\mathcal{L}_q$ and comparing equation (A.5) with equation (2.7), we prove equation (2.9).

Appendix B. Trace at integer *q*

In this appendix we give a proof of equation (5.4), in particular, we prove

$$\mathrm{tr}\big(\mathcal{L}_q^n\big) = \mathrm{tr}\big(\mathcal{M}_q^n\big), \tag{B.1}$$

for arbitrary positive integers q and n. For convenience, we first consider the problem in the full space. The left-hand side of equation (B.1) is defined by

$$\mathrm{tr}\big(\mathcal{L}_q^n\big) = \int_0^1 \mathcal{L}_q^n(x, x)\,\mathrm{d}x = \frac{1}{1 - 2^{-n}} \sum_{i=0}^{2^n-1} \Pi_{j=0}^{n-1} \left[\frac{1 - \cos(2^t 2\pi x_i)}{2}\right]^q, \tag{B.2}$$

where $x_i = i/(2^n - 1)$ are fixed points of the n-iteration of the Bernoulli map, while the right-hand side is given by

$$\mathrm{tr}\left(\mathcal{M}_q^n\right) = \sum_{j=-q}^{q} \langle \mathcal{E}_j^* | \mathcal{L}_q^{\dagger n} | \mathcal{E}_j \rangle, \tag{B.3}$$

where $\mathcal{E}_j(x) = \mathrm{e}^{\mathrm{i}2j\pi x}$ and $\mathcal{E}_j^* = \mathcal{E}_{-j}$. Note that there is no complex conjugation in our definition of inner product, i.e., $\langle f|g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$, and, in the full space, $\mathcal{M}_q$ as the restriction of $\mathcal{L}_q^\dagger$ within the invariant subspace spanned by $\{\mathcal{E}_j\}_{|j|\leqslant q}$ is a $(2q+1)\times(2q+1)$ matrix.

Because

$$[\mathcal{L}_q f](x) = 2\left[\frac{1-\cos(2\pi x)}{2}\right]^q f(2x), \tag{B.4}$$

we can rewrite equation (B.2) in a form similar to equation (B.3), i.e.,

$$\mathrm{tr}\left(\mathcal{L}_q^n\right) = \frac{1}{2^n - 1}\sum_{i=1}^{2^n-1}\left[\mathcal{L}_q^n \mathcal{E}_0\right](x_i) = \sum_{j=-q}^{q} \langle \mathcal{E}_{j(2^n-1)} | \mathcal{L}_q^n | \mathcal{E}_0 \rangle \tag{B.5}$$

$$= \sum_{j=-q}^{q} \langle \mathcal{E}_0 | \mathcal{L}_q^{\dagger n} | \mathcal{E}_{j(2^n-1)} \rangle. \tag{B.5}$$

In deriving equation (B.5) we have made use of the facts that $\left[\mathcal{L}_q^n \mathcal{E}_0\right](0) = 0$ and $\sum_{j=1}^{k} \mathcal{E}_m(j/k) = k$ (or 0) if m is (or is not) a multiple of k.

The action of $\mathcal{L}_q^\dagger$ in the basis $\{\mathcal{E}_j\}$ is given by

$$\mathcal{L}_q^\dagger \mathcal{E}_j = \sum_k \langle \mathcal{E}_k^* | \mathcal{L}_q^\dagger | \mathcal{E}_j \rangle \mathcal{E}_k \tag{B.6}$$

where

$$\langle \mathcal{E}_k^* | \mathcal{L}_q^\dagger | \mathcal{E}_j \rangle = \int_0^1 2\left[\frac{1-\cos(2\pi x)}{2}\right]^q \mathrm{e}^{\mathrm{i}(j-2k)2\pi x}\,\mathrm{d}x. \tag{B.7}$$

From equation (B.7) we can see immediately that, for arbitrary integer l,

$$\langle \mathcal{E}_{k+l}^* | \mathcal{L}_q^\dagger | \mathcal{E}_{j+2l} \rangle = \langle \mathcal{E}_k^* | \mathcal{L}_q^\dagger | \mathcal{E}_j \rangle, \tag{B.8a}$$

which can be readily generalized to

$$\langle \mathcal{E}_{k+l}^* | \mathcal{L}_q^{\dagger n} | \mathcal{E}_{j+2^n l} \rangle = \langle \mathcal{E}_k^* | \mathcal{L}_q^{\dagger n} | \mathcal{E}_j \rangle. \tag{B.8b}$$

From equation (B.8) we obtain our conclusion,

$$\mathrm{tr}\left(\mathcal{L}_q^n\right) = \sum_{j=-q}^{q} \langle \mathcal{E}_0^* | \mathcal{L}_q^{\dagger n} | \mathcal{E}_{j(2^n-1)} \rangle = \sum_{j=-q}^{q} \langle \mathcal{E}_{-j}^* | \mathcal{L}_q^{\dagger n} | \mathcal{E}_{-j} \rangle = \mathrm{tr}\left(\mathcal{M}_q^n\right).$$

Then we take the reflection symmetry into consideration. In the symmetric space, similar to equation (B.5), we have

$$\mathrm{tr}\left(\mathcal{L}_q^n\right) = \langle e_0 | \mathcal{L}_q^{\dagger n} \left| e_0 + \sum_{j=1}^{q} [e_{j(2^n-1)} + e_{j(2^n+1)}] \right\rangle. \tag{B.10}$$

Substituting e_j by $(\mathcal{E}_j + \mathcal{E}_{-j})/2$, we have

$$\begin{aligned}\operatorname{tr}\left(\mathcal{L}_q^n\right) &= \frac{1}{2}\sum_{j=-q}^{q}\langle\mathcal{E}_0^*|\mathcal{L}_q^{\dagger n}|[\mathcal{E}_{j(2^n-1)} + \mathcal{E}_{j(2^n+1)}]\rangle \\ &= \frac{1}{2}\sum_{j=-q}^{q}\langle\mathcal{E}_{-j}^*|\mathcal{L}_q^{\dagger n}|[\mathcal{E}_{-j} + \mathcal{E}_j]\rangle \\ &= \frac{1}{4}\sum_{j=-q}^{q}\langle[\mathcal{E}_{-j}^* + \mathcal{E}_j^*]|\mathcal{L}_q^{\dagger n}|[\mathcal{E}_{-j} + \mathcal{E}_j]\rangle \\ &= \sum_{j=0}^{q}\langle e_j^*|\mathcal{L}_q^{\dagger n}|e_j\rangle = \operatorname{tr}\left(\mathcal{M}_q^n\right).\end{aligned} \tag{B.11}$$

Finally, we explain why the reasoning in section 5 is insufficient to justify that $\det(1-z\mathcal{L}_q)$ is a polynomial. Let us consider a counterexample. Since the same reasoning can be applied to the case when $A(x)$ is replaced by

$$A(x,\eta) = \log\left[\frac{1-\eta\cos 2\pi x}{2}\right] \qquad (|\eta| \leqslant 1), \tag{B.12}$$

one may infer that $\det(1 - z\mathcal{L}_q) = \det(1 - z\mathcal{M}_q)$ also holds in this case. However, it can be easily shown that this is not true. In fact, since $\mathrm{e}^{A(0)} \neq 0$, we have

$$\operatorname{tr}\left(\mathcal{L}_q^n\right) = \operatorname{tr}\left(\mathcal{M}_q^n\right) + \frac{\mathrm{e}^{qnA(0)}}{1-2^{-n}} = \operatorname{tr}\left(\mathcal{M}_q^n\right) + \sum_{j=0}^{\infty}\left[\frac{(1-\eta)^q}{2^{q+j}}\right]^n \tag{B.13a}$$

in the full space and

$$\operatorname{tr}\left(\mathcal{L}_q^n\right) = \operatorname{tr}\left(\mathcal{M}_q^n\right) + \frac{\mathrm{e}^{qnA(0)}}{2^n-2^{-n}} = \operatorname{tr}\left(\mathcal{M}_q^n\right) + \sum_{j=0}^{\infty}\left[\frac{(1-\eta)^q}{2^{q+2j+1}}\right]^n \tag{B.13b}$$

in the symmetric space. Consequently,

$$\det(1 - z\mathcal{L}_q) = \det(1 - z\mathcal{M}_q)\Pi_{j=0}^{\infty}\left(1 - z\frac{(1-\eta)^q}{2^{q+j}}\right) \tag{B.14a}$$

and

$$\det(1 - z\mathcal{L}_q) = \det(1 - z\mathcal{M}_q)\Pi_{j=0}^{\infty}\left(1 - z\frac{(1-\eta)^q}{2^{q+2j+1}}\right) \tag{B.14b}$$

in the full and symmetric spaces, respectively. According to equation (B.14), the spectrum of $\mathcal{L}_q$ consists of two components, one comes from the finite-dimensional matrix $\mathcal{M}_q$ and the other is given by an infinite geometric series. The two parts of spectra correspond to eigenvectors in different function spaces, i.e., $C^\infty(S^1)$ for the former and $C^\infty([0, 1])$ for the latter.

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